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Nonnegative Fourier–Jacobi Coefficients and Some Classes of Functions

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Fourier–Jacobi series with nonnegative Fourier–Jacobi coefficients are considered. Under special restrictions on the Jacobi weight function, we establish in terms of Fourier–Jacobi coefficients a necessary and sufficient condition in order that the sum of the Fourier–Jacobi series should possess certain structural properties.

1. INTRODUCTION

The trigonometric series

$$f(x)=\sum_{n=0}^{\infty} a_n \cos nx, \qquad a_n \downarrow 0, \tag{1}$$

and

$$g(x)=\sum_{n=0}^{\infty} a_n \sin nx, \qquad a_n \downarrow 0, \tag{2}$$

have attracted mathematicians' attention for a long time. The first results in this area belong to Fatou [3, 4]. It is easy to prove by applying Abel's transformation that the series (1) and (2) converge uniformly on any interval $[\delta, \pi-\delta]$, $0<\delta<\pi$. In [2] a necessary and sufficient condition is given for the uniform convergence of the series (2) on $[0, 2\pi]$. Making use of this result, it is easy to give an example of a trigonometric series that converges uniformly but not absolutely on $[0, 2\pi]$; such a series is $\sum_{n=2}^{\infty} \sin nx/n \ln n$.

The interest in the investigation of the series (1) and (2) can be explained in two ways. First of all if one wishes to prove a statement concerning general trigonometric series, then very often it is helpful to have at one's

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disposal the same statement proved for a sufficiently large number of particular cases. On the other hand, some series of type (1) and (2) play an important role in the general theory of trigonometric series. For example, when dealing with the Gibbs phenomenon for functions of bounded variation, it is essential to know the behavior of the partial sums of the series $\sum_{n=1}^{\infty} \sin nx/n$ in a neighborhood of $x=0$. Besides, these partial sums are useful in the construction of Fejér's examples of functions whose Fourier series converge everywhere on $[0, 2\pi]$ but do not do so uniformly. We note also that the necessary and sufficient condition for the function (1) or (2) to belong to the space $L^p([0, 2\pi])$, $p>1$, plays a significant role in obtaining the main results of [6, 7] (the condition is $\sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty$).

We will write $f \in \operatorname{Lip} \gamma$ $(0<\gamma\leqslant 1)$ on $[a, b]$ if there is a constant $M>0$ such that $\forall x_1, x_2 \in [a, b]$ the inequality

$$|f(x_2)-f(x_1)| \leqslant M \cdot |x_2-x_1|^{\gamma}$$

holds and, moreover, M does not depend on x_1, x_2. In [9], Lorentz has found a necessary and sufficient condition for a function of type (1) or (2) to belong to the class $\operatorname{Lip} \gamma$ $(0<\gamma<1)$ on $[0, 2\pi]$.

In this paper we will prove a similar statement for the Fourier–Jacobi series.

2. NOTATION

Let C be the space of functions that are continuous on $[-1, 1]$, $\|f\|_C = \|f\| = \max\{|f(x)| : |x| \leqslant 1\}$, and $L_{\alpha, \beta}$ $(\alpha, \beta > -1)$ be the space of functions that are Lebesgue integrable on $[-1, 1]$ with the weight function $(1-x)^{\alpha}(1+x)^{\beta}$. By N we denote the set of all natural numbers and by W the set of all nonnegative integers. H_n $(n \in W)$ is the set of all algebraic polynomials of degree at most n. For $f \in C$, $n \in W$, we set

$$E_n(f)_C = E_n(f) = \inf\{\|f-Q_n\|_C : Q \in H_n\}.$$

$E_n(f)$ is the best approximation of f in the C metric by algebraic polynomials of degree at most n. $\{J_n^{(\alpha, \beta)}\}_0^{\infty} = \{J_n\}_0^{\infty}$ is the system of Jacobi polynomials, orthonormal on $[-1, 1]$ with the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ $(\alpha, \beta > -1)$, $J_n(1)>0$ $\forall n \in W$. We denote by $a_n^{(\alpha, \beta)}(f) = a_n(f)$ $(n \in W)$ the Fourier–Jacobi coefficients of $f \in L_{\alpha, \beta}$. By $S_n^{(\alpha, \beta)}(f) = S_n(f)$ $(n \in W)$ we denote the nth partial sum of the Fourier–Jacobi series $\sum_{n=0}^{\infty} a_n(f) J_n$. For $a \in R$ the symbol $[a]$ denotes the greatest integer not exceeding a. By A and by A with arguments between parentheses we denote, (in general, different) absolute positive constants and positive constants depending on the corresponding arguments, respectively. For two sequences $\{\alpha_n\}_0^\infty$, $\{\beta_n\}_0^\infty$ of positive numbers we will write $\alpha_n \sim \beta_n$ if there exist constants $A_1, A_2 > 0$, independent of n, such that

$$A_1 \beta_n \leqslant \alpha_n \leqslant A_2 \beta_n \qquad (n \in W).$$

Γ denotes Euler's Gamma function.

3. THE MAIN RESULTS

The primary purpose of this paper is to prove the following.

THEOREM. *Let* $-1/2 \leqslant \beta \leqslant \alpha < 1/2$ *or* $-1/2 \leqslant \alpha \leqslant \beta < 1/2$ *and assume that* $f \in C$, $a_n(f) \geqslant 0$ $\forall n \in W$. *Then*

$$f = \sum_{n=0}^{\infty} a_n(f)\, J_n \qquad \text{in the } C \text{ metric}; \tag{3}$$

in addition, for the relations

$$f^{(m)} = \sum_{n=m}^{\infty} a_n(f)\, J_n^{(m)} \quad (m \in W) \qquad \text{in the } C \text{ metric} \tag{4}$$

and

$$f^{(m)} \in \operatorname{Lip} \gamma, \qquad 0 < \gamma \leqslant 1, \tag{5}$$

to be valid simultaneously, it is necessary and sufficient that the inequality

$$\sum_{k=m}^{n} a_k(f)\, k^{5/2 + 2m + \sigma} \leqslant A(m, \alpha, \beta) \cdot (n+1)^{2 - 2\gamma} \tag{6}$$

should hold, where $\sigma = \max\{\alpha, \beta\}$, $n, n - m \in W$.

Before we proceed to prove the theorem, we mention two papers dealing with the same type of problems. In [10] one has established in terms of the sequence $\{c_n\}_0^\infty$ a sufficient condition for the function $\sum_{n=0}^{\infty} c_n J_n^{(\alpha, \beta)}$ $(\alpha, \beta > -1)$ to have r continuous derivatives on $(-1, 1)$. In [8] the author has imposed on the Fourier–Jacobi coefficients special monotonicity conditions and has established, under certain restrictions on α and β, a necessary and sufficient condition in terms of the Fourier–Jacobi coefficients in order that the Fourier–Jacobi series be convergent in the $L_{\alpha, \beta}$ metric.

4. PRELIMINARY LEMMAS

We will use the following formula [12, formulas (4.21.7) and (4.3.4)]

$$(J_n^{(\alpha,\beta)})^{(m)} = C_n(\alpha, \beta, m)\, J_{n-m}^{(\alpha+m,\beta+m)}, \qquad n, m-m \in W, \tag{7}$$

where

$$C_n(\alpha, \beta, m) = (\Gamma(n+1)\,\Gamma(n+\alpha+\beta+m+1)\,\Gamma^{-1}(n-m+1)\,\Gamma^{-1}(n+\alpha+\beta+1))^{1/2};$$

for large n we have

$$C_n(\alpha, \beta, m) \sim n^m. \tag{8}$$

We will also make use of the following estimate [12, formula (7.32.2)]:

$$\|J_n\| \leqslant A(\alpha, \beta)(n+1)^{\sigma+1/2}, \qquad n \in W, \quad \sigma = \max\{\alpha, \beta\} \geqslant -\tfrac{1}{2}. \tag{9}$$

LEMMA 1. *Let* $-1/2 \leqslant \alpha, \beta \leqslant 1/2$, $k \in W$, $0 \leqslant (2k+1)\, t \leqslant \pi$. *Then*

$$J_k(1) - J_k(\cos t) \geqslant A(\alpha, \beta)\, k^{\alpha+5/2} t^2. \tag{10}$$

Proof. Let x_1 be the largest zero of J_k $(k \in N)$, $x_1 = \cos \varphi_1$, $0 < \varphi < \pi$. Since $-1/2 \leqslant \alpha, \beta \leqslant 1/2$, we have $\varphi_1 \geqslant \pi/(2k+1)$ [12, formula (6.21.5)] and, therefore, the inequality $(2k+1)\, t \leqslant \pi$ implies $\cos t \geqslant x_1$. Further,

$$J_k(1) - J_k(\cos t) = \int_{\cos t}^{1} J'_k(z)\, dz \geqslant J'_k(x_1)(1-\cos t). \tag{11}$$

As proved in [13], we have

$$J'_k(x_1) \geqslant A(\alpha, \beta)\, k^{\alpha+5/2}. \tag{12}$$

From (11) and (12) we obtain at once (10). Lemma 1 is proved.

For $f \in C$ we introduce $(C, 1)$-sums with respect to the system $\{J_n\}_0^\infty$:

$$\sigma_n^{(\alpha,\beta)}(f) = \sigma_n(f) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k(f), \qquad n \in W.$$

We need the following proposition [1, p. 32].

PROPOSITION. *If $\beta \leqslant \alpha < 1/2$, then*

$$\|\sigma_n\|_{C \to C} \leqslant A(\alpha, \beta), \qquad n \in W. \tag{13}$$

For $\alpha = \beta = 0$, inequality (13) *has been proved in* [5].

LEMMA 2. *Let $\beta \leqslant \alpha < 1/2$, $f \in C$, $a_n(f) \geqslant 0$ $\forall n \in W$. Then*

$$\|S_n(f)\| \leqslant \|f\|, \qquad n \in W. \tag{14}$$

Proof. We may assume that $n \geqslant 1$ since inequality (14) is trivial for $n = 0$. We introduce the de la Vallée–Poussin sums for f by

$$\tau_n^{(\alpha, \beta)}(f) = \tau_n(f) = \frac{1}{n} \sum_{k=n}^{2n-1} S_k(f) = 2\sigma_{2n-1}(f) - \sigma_{n-1}(f). \tag{15}$$

By virtue of (15) and (13), $\forall f \in C$, $\forall n \in W$ we have

$$\|\tau_n(f)\| \leqslant A(\alpha, \beta)\, \|f\|. \tag{16}$$

Let $Q_m \in H_m$ $(m \in N)$, $\|f - Q_m\| = E_m(f)$; since $\tau_m(Q_m) = Q_m$, in view of (16) we obtain

$$\|f - \tau_m(f)\| \leqslant \|f - Q_m\| + \|\tau_m(f - Q_m)\| \leqslant A(\alpha, \beta)\, E_m(f). \tag{17}$$

We will make use of the following easily verifiable equality:

$$\tau_m(f) = S_m(f) + \frac{1}{m} \sum_{k=1}^{m-1} (m-k)\, a_{m+k}(f)\, J_{m+k}.$$

If $m \geqslant n$, then we have

$$\begin{aligned} \|S_n(f)\| &= S_n(f; 1) \leqslant S_m(f; 1) \\ &= \tau_m(f; 1) - \frac{1}{m} \sum_{k=1}^{m-1} (m-k)\, a_{m+k}(f)\, J_{m+k}(1) \\ &\leqslant \tau_m(f; 1) = \|\tau_m(f)\|. \end{aligned} \tag{18}$$

If in (18) we let $m \to \infty$ and we take into account (17), then we obtain (14). Lemma 2 is proved.

LEMMA 3. *Under the conditions of Lemma 2, we have*

$$\|f - S_n(f)\| \leqslant A(\alpha, \beta)\, E_{[n/2]}(f). \tag{19}$$

Proof. We may assume that $n \geqslant 1$ since inequality (19) is trivial for $n = 0$. Since $\tau_{[n/2]}(f) \in H_n$, we have

$$f - S_n(f) = f - \tau_{[n/2]}(f) + S_n(\tau_{[n/2]}(f) - f)$$

and, consequently, we obtain

$$\|f - S_n(f)\| \leqslant \|f - \tau_{[n/2]}(f)\| + \|S_n(\tau_{[n/2]}(f) - f)\|. \tag{20}$$

Making use of (17) we derive

$$\|f - \tau_{[n/2]}(f)\| \leqslant A(\alpha, \beta)\, E_{[n/2]}(f). \tag{21}$$

Taking into account (15), one can easily verify that

$$\tau_n(f) = 2 \sum_{k=n}^{2n-1} \left(1 - \frac{k}{2n}\right) a_k(f)\, J_k + \sum_{k=0}^{n-1} a_k(f)\, J_k. \tag{22}$$

It follows from (27) that

$$a_k(f - \tau_{[n/2]}(f)) = \begin{cases} 0, & k \leqslant \left[\dfrac{n}{2}\right] \\ \left(\dfrac{k}{[n/2]} - 1\right) a_k(f), & \left[\dfrac{n}{2}\right] < k \leqslant 2\left[\dfrac{n}{2}\right] - 1, \\ a_k(f), & k \geqslant 2\left[\dfrac{n}{2}\right] \end{cases}$$

which implies that

$$a_k(f - \tau_{[n/2]}(f)) \geqslant 0, \qquad k \in W. \tag{23}$$

In view of (23) we can apply (14) to $f - \tau_{[n/2]}(f)$:

$$\|S_n(f - \tau_{[n/2]}(f))\| \leqslant \|f - \tau_{[n/2]}(f)\| \leqslant A(\alpha, \beta)\, E_{[n/2]}(f). \tag{24}$$

Combining (20),(21), and (24), we obtain (19). Lemma 3 is proved.

Remark. For $\alpha = \beta = 0$, Lemmas 2 and 3 have been proved in [11].

5. PROOF OF THE THEOREM.

We are now in position to prove the theorem stated in Section 3. Assume that $-1/2 \leqslant \beta \leqslant \alpha < 1/2$. First we note that the expansion (3) follows directly from (19). We will prove now that if the estimate (6) holds, then the relations (4) and (5) are simultaneously valid. First we prove that

$$\sum_{k=m}^{\infty} a_k(f)(k+1)^{2m+\alpha+1/2} < \infty. \tag{25}$$

We introduce

$$\rho_n = \sum_{k=m}^{n} a_k(f)(k+1)^{2m+\alpha+5/2}, \quad m \geqslant m; \qquad \rho_{m-1} = 0.$$

Making use of Abel's transformation, we obtain

$$\sum_{k=m}^{\infty} a_k(f)(k+1)^{2m+\alpha+1/2} = \sum_{k=m}^{\infty} \rho_k((k+1)^{-2} - (k+2)^{-2})$$
$$\leqslant A(m, \alpha, \beta) \sum_{k=m}^{\infty} (k+1)^{-1-2\gamma} < \infty.$$

Differentiating formally the series (3) m times, we obtain the series $\sum_{n=m}^{\infty} a_n(f)\, J_n^{(m)}$. Taking into consideration (7)–(9) and (25), we obtain

$$\sum_{k=m}^{\infty} a_k(f)\, \|J_k^{(m)}\| \leqslant A(\alpha, \beta, m) \sum_{k=m}^{\infty} a_k(f)(k+1)^{\alpha+2m+1/2} < \infty$$

and, consequently, $f^{(m)} \in C$ and we have the following Fourier–Jacobi series expansion in the C metric:

$$f^{(m)} = \sum_{k=m}^{\infty} a_k(f)(J_k)^{(m)} = \sum_{k=m}^{\infty} a_k(f)\, C_k(\alpha, \beta, m)\, J_{k-m}^{(\alpha+m,\, \beta+m)}.$$

Let $x_1, x_2 \in [-1, 1]$, $x_1 \neq x_2$. We have

$$f^{(m)}(x_2) - f^{(m)}(x_1)$$
$$= \sum_{k=m}^{\infty} a_k(f)\, C_k(\alpha, \beta, m)(J_{k-m}^{(\alpha+m,\, \beta+m)}(x_2) - J_{k-m}^{(\alpha+m,\, \beta+m)}(x_1)).$$

Assume that $M \in W$, $M \geqslant m$. Then

$$
\begin{aligned}
|f^{(m)}(x_2) - f^{(m)}(x_1)| \leqslant \sum_{k=m}^{M} & a_k(f)\, C_k(\alpha, \beta, m) \\
& \times |J_{k-m}^{(\alpha+m,\, \beta+m)}(x_2) - J_{k-m}^{(\alpha+m,\, \beta+m)}(x_1)| \\
& + 2 \sum_{k=M+1}^{\infty} a_k(f)\, C_k(\alpha, \beta, m)\, \|J_{k-m}^{(\alpha+m,\, \beta+m)}\|_C \\
\leqslant & A(m, \alpha, \beta)\, |x_2 - x_1| \cdot \sum_{k=m}^{M} a_k(f)\, k^m\, \|(J_{k-m}^{(\alpha+m,\, \beta+m)})'\| \\
& + A(m, \alpha, \beta) \sum_{k=M+1}^{\infty} a_k(f)\, k^{\alpha+2m+1/2} \\
= & S_1 + S_2. \qquad (26)
\end{aligned}
$$

S_2 has been estimated above:

$$S_2 \leqslant A(m, \alpha, \beta) \cdot (M+1)^{-2\gamma}. \qquad (27)$$

Making use of (7)–(9) and (6), we obtain

$$S_1 \leqslant A(m, \alpha, \beta) \cdot |x_2 - x_1| \cdot (M+1)^{2-6\gamma}. \qquad (28)$$

It follows from (26)–(28) that

$$
\begin{aligned}
& |f^{(m)}(x_2) - f^{(m)}(x_1)| \\
& \quad \leqslant A(m, \alpha, \beta)((M+1)^{-2\gamma} + |x_2 - x_1|\, (M+1)^{2-2\gamma}). \qquad (29)
\end{aligned}
$$

We set

$$M + 1 = [m + |x_2 - x_1|^{-1/2} + 2].$$

It is obvious that

$$|x_2 - x_1|^{-1/2} \leqslant M + 1 \leqslant A(m)\, |x_2 - x_1|^{-1/2}. \qquad (30)$$

Combining the estimates (29) and (30), we derive that

$$|f^{(m)}(x_2) - f^{(m)}(x_1)| \leqslant A(m, \alpha, \beta) \cdot |x_2 - x_1|^{\gamma},$$

i.e., $f^{(m)} \in \operatorname{Lip} \gamma$ on, $[-1, 1]$.

It remains to prove that if the relations (4) and (5) hold simultaneously, then inequality (6) is valid. We have

$$\left|\sum_{k=m}^{\infty} a_k(f)[J_k^{(m)}(1)-J_k^{(m)}(\cos t)]\right| = |f^{(m)}(1)-f^{(m)}(\cos t)|$$
$$\leqslant A\cdot(1-\cos t)^{\gamma} \leqslant A\cdot t^{2\gamma},$$
$$0 \leqslant t \leqslant \pi. \tag{31}$$

Making use of Lemma 1, we obtain

$$\left|\sum_{k=m}^{\infty} a_k(f)[J_k^{(m)}(1)-J_k^{(m)}(\cos t)]\right|$$
$$\geqslant A(m,\alpha,\beta) \sum_{k=m}^{[\pi(2t)^{-1}+m-2^{-1}]} a_k(f)\cdot k^m$$
$$\times [J_{k-m}^{(\alpha+m,\,\beta+m)}(1)-J_{k-m}^{(\alpha+m,\,\beta+m)}(\cos t)]$$
$$\geqslant A(m,\alpha,\beta)\, t^2 \sum_{k=m}^{[\pi\cdot(2t)^{-1}+m-2^{-1}]} a_k(f)\cdot k^{\alpha+2m+5/2}. \tag{32}$$

It follows from (31) and (32) that

$$\sum_{k=m}^{[\pi(2t)^{-1}+m-1/2]} a_k(f)\cdot k^{\alpha+2m+5/2} \leqslant A(m,\alpha,\beta)\, t^{2\gamma-2}. \tag{33}$$

Setting $t=\pi/(2n-2m+1)$ in (33), we obtain

$$\sum_{k=m}^{n} a_k(f)\cdot k^{\alpha+2m+5/2} \leqslant A(m,\alpha,\beta)(n+1)^{2-2\gamma}.$$

This completes the proof of the theorem in the case $-1/2 \leqslant \beta \leqslant \alpha < 1/2$. The case $-1/2 \leqslant \alpha \leqslant \beta < 1/2$ can be easily reduced to the first one if (i) we apply the assertions of the theorem just proved to the function $\varphi(x)=f(-x)$ and (ii) we take into account that $\forall n \in W$, $\forall x \in R$ we have $J_n^{(\beta,\alpha)}(-x)=(-1)^n J_n^{(\alpha,\beta)}(x)$.

COROLLARY. *Let* $f \in C$, $a_n(f) \downarrow 0$. *Then* (i) *the relation* (3) *holds*; (ii) *the relations* (4) *and* (5) *are simultaneously valid if and only if* $\forall n \in W$ *we have*

$$a_n(f) \leqslant A(m,\alpha,\beta)\cdot(n+1)^{-3/2-2m-\sigma-2\gamma}.$$

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